Objectives:

- Define and use the Mean Value Theorem.

Suppose you drive from Boulder to Denver along I-25 and US36. You decide to drive in the toll lane for 23 miles. Two weeks later you receive a speeding ticket in the mail. The ticket claims that you entered the toll road at 4:45 pm and exited 23 miles later at 5:00 pm. If you decide to appeal the ticket, can they prove you were speeding?

Your average speed was 23 miles per 15 min, or 23/.25=92 miles per hour. So it seems reasonable to assume you were speeding at some point! But can they prove it??

The Mean Value Theorem (MVT):

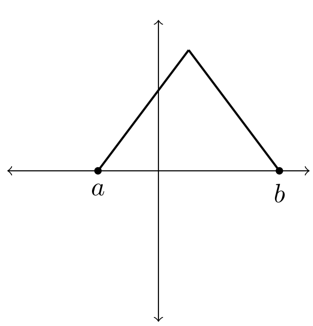
If f is continuous on a closed interval, $[a, b]$, and f is differentiable on (a,b) , then there exists a number c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In words, this means that if f meets these requirements over an interval $[a, b]$, then there is a point in the interval where the instantaneous rate of change (derivative) at that point is equal to the average rate of change of the interval.

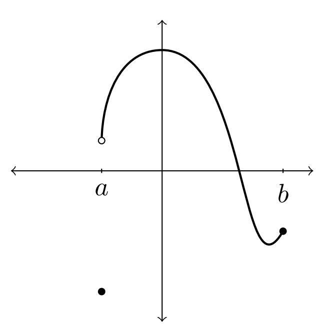
Graphically, this means that if f satisfies these hypotheses over $[a, b]$, then some point in the interval has a tangent line that is parallel to the secant line between $(a, f(a))$ and $(b, f(b))$

For the following functions, are the hypotheses of the Mean Value Theorem true? Is the conclusion of the Mean Value Theorem true?



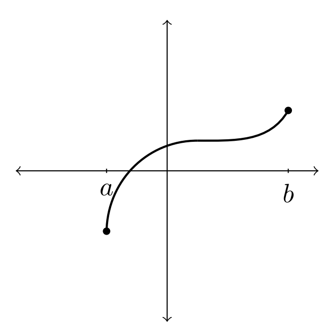
Hypotheses: Not differentiable on (a, b)

Conlcusion: No. Secant line slope= 0. No point has horizontal tangent.



Hypotheses: Not continuous on $[a, b]$

Conclusion: True! (Draw secant and tangent lines)



Hypotheses: True

Conclusion: Must be true by MVT. (There are even 2 such pts)

Back to our example: We could represent the position of the car, $s(t)$ in miles, as a function of time, t in minutes after 4:45pm.

Does this function on the interval $[0, 15]$ meet the hypotheses of the Mean Value Theorem?
Yes. We can assume the position of the car is continuous (no teleportation) and that the velocity is defined always (except perhaps at endpoints).

What does the Mean Value Theorem imply in this case?

Average velocity is $\dfrac{s(15)-s(0)}{15-0} = \dfrac{23}{15}\dfrac{\text{miles}}{\text{minute}}$ or $\dfrac{23}{.25} = 92$ mph. So , there must be some point in time in the interval (i.e. while you were on the highway) where the instantaneous velocity was 92 mph. So, they could definitely prove that you were speeding.

In general, how does the Mean Value Theorem apply to velocity?

When the position function of an object is continuous and differentiable on an interval, there is some point in the interval where the instantaneous velocity ("spedometer reading") is equal to the average velocity over the interval.

More Examples: What does the MVT tell us about the following functions?

1. $f(x) = x^2$ on $[1, 3]$

   Continuous, check. Differentiable, check. So, by MVT, there is some $1 \leq c \leq 3$ such that $f'(c) = \dfrac{f(3)-f(1)}{3-1} = \dfrac{9-1}{2} = 4$.
   In fact, we can find such a c: $f'(x) = 2x$ so if $f'(c) = 4$ then $4 = 2c$ and thus $c = 2$.

2. $g(t) = \dfrac{1}{t}$ on $[-1, 1]$

   $g(t)$ is NOT continuous on $[-1, 1]$, so MVT tells us nothing.
   $\dfrac{g(1)-g(-1)}{1-(-1)} = \dfrac{1-(-1)}{1-(-1)} = 1$ but $g'(t) = \dfrac{-1}{x^2}$ which is always negative. So, in this case, there is no c s.t. $g'(c) = 1$.

Example: Prove $f(x) = x^3 + x + 1$ has only one zero.

$f(x)$ is continuous everywhere. Since $f(-1) < 0$ and $f(1) > 0$, we can then use the Intermediate Value Theorem to say there's a zero in the interval $[-1, 1]$. Call this zero $x = a$
What happens if there's another zero, $x = b \neq a$? Then, since $f(x)$ is continuous and differentiable, there exists some c in between a and b such that $f'(c) = \dfrac{f(a)-f(b)}{a-b} = \dfrac{0-0}{a-b} = 0$. But $f'(x) = 3x^2+1$ is never 0. So it is impossible that $f(x)$ has another zero.